Tutorial 5

Simplex method

Simplex method is a method to solve the linear programming problems.

Given an $m \times n$ matrix A, two vectors $\boldsymbol{b} \in \mathcal{P}^m$, $\boldsymbol{c} \in \mathcal{P}^n$ and a number d, we consider *primal problem*

$$\begin{aligned} \max \quad & f(\boldsymbol{y}) = \boldsymbol{c}\boldsymbol{y}^T + d \\ \text{subject to} \quad & A\boldsymbol{y}^T \le \boldsymbol{b}^T \end{aligned}$$

and the *dual problem*

$$\begin{aligned} \min \quad & g(\boldsymbol{x}) = \boldsymbol{x}\boldsymbol{b}^T + d \\ \text{subject to} \quad & \boldsymbol{x}A \ge \boldsymbol{c}. \end{aligned}$$

The key step of the simplex method is called the **pivoting operation**. Assume the tableau of the linear programming problem is given by

	y_1	$\cdots$	y_n	-1
x_1	a_{11}	$\cdots$	a_{1n}	b_1
$\vdots$	$\vdots$		$\vdots$	$\vdots$
x_m	a_{m1}	$\cdots$	a_{mn}	b_m
-1	c_1	$\cdots$	c_n	$-d$

Step 1. Find a position to start the pivoting operation.

If $c_j \le 0$ for all j, then go to step 3. Otherwise, choose $j \in \{1, 2, \cdots, n\}$ such that $c_j > 0$.

If $a_{ij} \le 0$ for all $1 \le i \le m$, the primal problem has no solution. Otherwise, pick $k \in \{1, 2, \cdots, m\}$ such that

$$\frac{b_k}{a_{kj}} = \min\{\frac{b_i}{a_{ij}} : a_{ij} > 0, i = 1, \cdots, m\}.$$

Step 2. Make pivoting operation as follows.

	y_k	y_l	
x_i	a^*	b	
x_j	c	d	

$$\longrightarrow$$

	x_i	y_l	
y_k	$\frac{1}{a}$	$\frac{b}{a}$	
x_j	$-\frac{c}{a}$	$\frac{ad-bc}{a}$	

Step 3. Continue Step 1 and Step 2 until $c_j \leq 0$ for all j. If the final result after pivoting operations is

	x_i	y_l	-1
y_j			e
x_k			f
-1	g	h	$-v$

,

then we can conclude that the optimal value of the primal problem is v and

$$x_i = -g \quad y_{i+n} = 0$$
$$y_l = 0 \quad x_{l+m} = -h$$
$$y_j = e \quad x_{j+n} = 0$$
$$x_k = 0 \quad y_{k+m} = f.$$

Exercise 1. *Use the simplex method to solve the two-person zero-sum game* *with game matrix*

$$\begin{pmatrix} -1 & 1 & 3 \\ 1 & -3 & 2 \\ 3 & 0 & -1 \end{pmatrix}.$$

Solution. Step 1. Add 3 to each entry, we get

$$\begin{pmatrix} 2 & 4 & 6 \\ 4 & 0 & 5 \\ 6 & 3 & 2 \end{pmatrix}.$$

Step 2. Set up the tableau as

$$\begin{array}{c|ccc|c} & y_1 & y_2 & y_3 & -1 \\ \hline x_1 & 2 & 4 & 6 & 1 \\ x_2 & 4 & 0 & 5 & 1 \\ x_3 & 6 & 3 & 2 & 1 \\ \hline -1 & 1 & 1 & 1 & 0 \end{array}.$$

Step 3. Apply pivoting operations, we have

$$\begin{array}{c|ccc|c} & y_1 & y_2 & y_3 & -1 \\ \hline x_1 & 2 & 4 & 6 & 1 \\ x_2 & 4 & 0 & 5 & 1 \\ x_3 & 6^* & 3 & 2 & 1 \\ \hline -1 & 1 & 1 & 1 & 0 \end{array} \rightarrow \begin{array}{c|ccc|c} & x_3 & y_2 & y_3 & -1 \\ \hline x_1 & -\frac{1}{3} & 3^* & \frac{16}{3} & \frac{2}{3} \\ x_2 & -\frac{2}{3} & -2 & \frac{11}{3} & \frac{1}{3} \\ y_1 & \frac{1}{6} & \frac{1}{2} & \frac{1}{3} & \frac{1}{6} \\ \hline -1 & -\frac{1}{6} & \frac{1}{2} & \frac{2}{3} & -\frac{1}{6} \end{array} \rightarrow$$

	x_3	x_1	y_3	-1
y_2	$-\frac{1}{9}$	$\frac{1}{3}$	$\frac{16}{9}$	$\frac{2}{9}$
$\rightarrow$ x_2	$-\frac{8}{9}$	$\frac{2}{3}$	$\frac{115}{27}$	$\frac{11}{27}$
y_1	$\frac{2}{9}$	$-\frac{1}{6}$	$-\frac{5}{9}$	$\frac{1}{18}$
-1	$-\frac{1}{9}$	$-\frac{1}{6}$	$-\frac{2}{9}$	$-\frac{5}{18}$

.

Let $d = \frac{5}{18}$. Then the value of the game is $v = \frac{1}{d} - 3 = \frac{3}{5}$. Since the basic solution is

$$\begin{aligned} x_3 &= \frac{1}{9} & y_6 &= 0 \\ x_1 &= \frac{1}{6} & y_4 &= 0 \\ y_3 &= 0 & x_6 &= \frac{2}{9} \\ y_2 &= \frac{2}{9} & x_5 &= 0 \\ x_2 &= 0 & y_5 &= \frac{11}{27} \\ y_1 &= \frac{1}{18} & x_4 &= 0 \end{aligned}$$

We have the maximin strategy for the row player is

$$\boldsymbol{p} = \frac{1}{d}(x_1, x_2, x_3) = \frac{18}{5}(\frac{1}{6}, 0, \frac{1}{9}) = (\frac{3}{5}, 0, \frac{2}{5}),$$

and the minimax strategy for the column player is

$$\boldsymbol{q} = \frac{1}{d}(y_1, y_2, y_3) = \frac{18}{5}(\frac{1}{18}, \frac{2}{9}, 0) = (\frac{1}{5}, \frac{4}{5}, 0).$$

Exercise 2. *Let A be an $m \times n$ matrix. Let*

$$C = \mathrm{conv}(\{\mathrm{a_1}, \cdots, \mathrm{a_n}, \mathrm{e_1}, \cdots, \mathrm{e_m}\})$$

be the convex hull of set $\{\boldsymbol{a}_1, \cdots, \boldsymbol{a}_n, \boldsymbol{e}_1, \cdots, \boldsymbol{e}_m\}$*, where* $\boldsymbol{a}_1^T, \cdots, \boldsymbol{a}_n^T$ *are the column vectors of* A *and* $\boldsymbol{e}_1, \cdots, \boldsymbol{e}_m$ *are the vectors in the standard basis of* $\mathbb{R}^m$*. Prove if* C *contains a point* $(c, \cdots, c) \in \mathbb{R}^m$ *with* $c \leq 0$*, then the value of* A*,* $v(A) \leq c$*.*

Proof. Since $(c, \cdots, c) \in C$, there exist $\lambda_1, \cdots, \lambda_{n+m}$ such that

$$\lambda_1 \boldsymbol{a}_1 + \cdots + \lambda_n \boldsymbol{a}_n + \lambda_{n+1} \boldsymbol{e}_1 + \cdots + \lambda_{n+m} \boldsymbol{e}_m = (c, \cdots, c),$$

where $0 \leq \lambda_i \leq 1$ and $\lambda_1 + \cdots + \lambda_{n+m} = 1$.

Since $c \leq 0$, at least one of $\lambda_1, \cdots, \lambda_n$ is positive. Multiply both sides of the above equation by $\frac{1}{\lambda_1 + \cdots + \lambda_n}$, we have

$$\frac{\lambda_1}{\lambda_1 + \cdots + \lambda_n} \boldsymbol{a}_1 + \cdots + \frac{\lambda_n}{\lambda_1 + \cdots + \lambda_n} \boldsymbol{a}_n = \left(\frac{c - \lambda_{n+1}}{\lambda_1 + \cdots + \lambda_n}, \cdots, \frac{c - \lambda_{n+m}}{\lambda_1 + \cdots + \lambda_n}\right).$$

Taking transpose, we have

$$(\boldsymbol{a}_1^T \quad \cdots \quad \boldsymbol{a}_n^T) \begin{pmatrix} \frac{\lambda_1}{\lambda_1 + \cdots + \lambda_n} \\ \vdots \\ \frac{\lambda_n}{\lambda_1 + \cdots + \lambda_n} \end{pmatrix} = \frac{1}{\lambda_1 + \cdots + \lambda_n} \begin{pmatrix} c - \lambda_{n+1} \\ \cdots \\ c - \lambda_{n+m} \end{pmatrix}.$$

Note that $A = (\boldsymbol{a}_1^T, \cdots, \boldsymbol{a}_n^T)$. Write $\boldsymbol{q} = \frac{1}{\lambda_1 + \cdots + \lambda_n}(\lambda_1, \cdots, \lambda_n)$. Then $\boldsymbol{q} \in \mathcal{P}^n$ and

$$\boldsymbol{x} A \boldsymbol{q}^T \leq \max_{1 \leq i \leq m} \frac{c - \lambda_{n+i}}{\lambda_1 + \cdots + \lambda_n} \leq c, \text{ since } c \leq 0.$$

Hence

$$v(A) = \min_{\boldsymbol{y} \in \mathcal{P}^n} \max_{\boldsymbol{x} \in \mathcal{P}^m} \boldsymbol{x} A \boldsymbol{y}^T \leq \max_{x \in \mathcal{P}^m} \boldsymbol{x} A \boldsymbol{q}^T \leq c.$$